On the L^p-estimates for Beurling-Ahlfors and Riesz transforms on Riemannian manifolds

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Abstract

In our previous papers [6, 9], we proved some martingale transform representation formulas for the Riesz transforms and the Beurling-Ahlfors transforms on complete Riemannian manifolds, and proved some explicit L^p-norm estimates for these operators on complete Riemannian manifolds with suitable curvature conditions. In this paper we correct a gap contained in [6, 9] and prove that the L^p-norm of the Riesz transforms $R_a(L) = \nabla(a - L)^{-1/2}$ can be explicitly bounded by $C(p^* - 1)^{3/2}$ if $Ric + \nabla^2\phi \geq -a$ for $a \geq 0$, and the L^p-norm of the Riesz transform $R_0(L) = \nabla(-L)^{-1/2}$ is bounded by $2(p^* - 1)$ if $Ric + \nabla^2\phi = 0$. We also prove that the L^p-norm estimates for the Beurling-Ahlfors transforms obtained in [9] remain valid. Moreover, we prove the time reversal martingale transform representation formulas for the Riesz transforms and the Beurling-Ahlfors transforms on complete Riemannian manifolds.

1 Introduction

In our previous paper [6], the author obtained a martingale transform representation formula for the Riesz transforms on complete Riemannian manifolds. More precisely, by the formula (24) in Theorem 3.2 in [6], the probabilistic representation formula of the Riesz transform $R_a(L) = \nabla(a - L)^{-1/2}$ acting on a nice function f was given by

$$-\frac{1}{2}R_a(L)f(x) = \lim_{y\to+\infty} E_y\left[\int_0^\tau e^{a(s-\tau)}M_\tau M_s^{-1}dQ_af(X_s, B_s)dB_s\middle| X_\tau = x\right].$$

Recently, R. Bañuelos and F. Baudoin [2] pointed out that, since $e^{-a\tau}M_\tau$ is not adapted to the filtration $\mathcal{F}_t = \sigma(X_s, B_s, s \leq t)$, the above probabilistic representation formula should

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be corrected as follows

$$-\frac{1}{2}R_a(L)f(x) = \lim_{y\to+\infty} E_y\left[e^{-a\tau}M_\tau\int_0^\tau e^{as}M_s^{-1}dQ_af(X_s,B_s)dB_s\middle| X_\tau = x\right]. \tag{1}$$

Indeed, a careful check of the original proof of the formula (24) in Theorem 3.8 in [6] indicates that the correct probabilistic representation formula of $R_a(L)f$ should be given by (1). See Section 2 below. By the above observation, R. Bañuelos and F. Baudoin [2] pointed out that there is a gap in the proof of the L^p-norm estimates of the Riesz transforms in [6] and they proved a new martingale inequality which can be used to correct this gap. In this paper, we correct the above gap and prove that the L^p-norm of the Riesz transform $R_a(L)$ is bounded above by $C(p^*-1)^{3/2}$ if $Ric+\nabla^2\phi\geq -a$ for $a\geq 0$, and the L^p-norm of the Riesz transform $R_0(L)$ is bounded by $2(p^*-1)$ if $Ric+\nabla^2\phi = 0$. See Theorem 2.4 below. We also correct the gap contained in [9] (due to the same reason as above) and prove that the main results on the L^p-norm estimates of the Beurling-Ahlfors transforms obtained in [9] remain valid. See Theorem 4.4 and Remark 4.5 below. Moreover, we prove the time reversal martingale transform representation formulas for the Riesz transforms and the Beurling-Ahlfors transforms on complete Riemannian manifolds.

2 Riesz transforms on functions

Let (M,g) be a complete Riemannian manifold, ∇ the gradient operator on M, Δ the Laplace-Beltrami operator on M. Let $\phi\in C^2(M)$, and $d\mu = e^{-\phi}dv$, where dv is the standard Riemannian volume measure on M. Let $L^2_0(M,\mu)=L^2(M,\mu)$ if $\mu(M)=\infty$, and $L^2_0(M,\mu)=\{f\in L^2(M,\mu): \int_M fd\mu = 0\}$ if $\mu(M)<\infty$.

Let $L=\Delta-\nabla\phi\cdot\nabla$. Let d be the exterior differential operator, d^*_ϕ be its L^2-adjoint with respect to the weighted volume measure $d\mu = e^{-\phi}dv$. Let $\Box_\phi = dd^*_\phi + d^*_\phi d$ be the Witten-Laplacian acting on forms over (M,g) with respect to the weighted volume measure $d\mu = e^{-\phi}dv$.

Let B_t be one dimensional Brownian motion on $\mathbb{R}$ starting from $B_0=y>0$ and with infinitesimal generator $\frac{1}{2}\frac{d^2}{dy^2}$. Let

$$\tau = \inf\{t>0: B_t = 0\}.$$

Let X_t be the L-diffusion process on M. Let Ric be the Ricci curvature on (M,g), $\nabla^2\phi$ be the Hessian of the potential function ϕ. Let $M_t\in \mathrm{End}(T_{X_0}M, T_{X_t}M)$ is the unique solution to the covraiant SDE along the trajectory of (X_t):

$$\frac{\nabla}{\partial t}M_t = -(Ric+\nabla^2\phi)(X_t)M_t, \quad M_0 = \mathrm{Id}_{T_{X_0}M}.$$

In particular, in the case where $Ric+\nabla^2\phi=-a$, we have

$$M_t = e^{at}U_t, \quad \forall t\geq 0,$$

where $U_t: T_{X_0}M\to T_{X_t}M$ denotes the stochastic parallel transport along X_t.

The following result is the correct reformulation of Lemma 3.7 in [6].

Lemma 2.1 *For all $\eta \in C_0^\infty(M, \Lambda^1 T^*M)$, and $\eta_a(x,y) = e^{-y\sqrt{a+\square_\phi}}\eta(x)$, we have*

$$\eta(X_\tau) = e^{a\tau} M_{\tau,k}^{*,-1}\eta_a(X_0, B_0) + e^{a\tau} M_{\tau,k}^* \int_0^\tau e^{-as} M_s^* \left(\nabla, \frac{\partial}{\partial y}\right) \eta_a(X_s, B_s) \cdot (U_s dW_s, dB_s). \quad (2)$$

Proof. By Itô's calculus, we have (see p.266 line 16 in [6])

$$\frac{\nabla}{\partial t}\left(e^{-at} M_t^* \eta_a(X_t, B_t)\right) = e^{-at} M_t^* \left(\nabla, \frac{\partial}{\partial y}\right) \eta_a(X_t, B_t) \cdot (U_t dW_t, dB_t).$$

Integrating from $t = 0$ to $t = \tau$, we complete the proof of Lemma 2.1. □

The following result is the correct reformulation of Theorem 3.8 in [6].

Theorem 2.2 *Let $\omega \in C_0^\infty(M < \Lambda^1 T^*M)$, and $\omega_a(x,y) = e^{-y\sqrt{a+\square_\phi}}\omega(x)$. Then*

$$\frac{1}{2}\omega(x) = \lim_{y\to\infty} E_y \left[e^{-a\tau} M_\tau \int_0^\tau e^{as} M_s^{-1} \frac{\partial}{\partial y}\omega_a(X_s, B_s) dB_s \middle| X_\tau = x \right]. \quad (3)$$

Proof. The proof is indeed a small modification of the original proof of Theorem 3.8 given in [6]. For the completeness of the paper, we produce the details here. Let $Z_t = (X_t, B_t)$, $\eta \in C_0^\infty(\Lambda^k T^*M)$. By (2) in Lemma 2.1, we have

$$\eta(X_\tau) = e^{a\tau} M_\tau^{*,-1}\eta_a(Z_0) + e^{a\tau} M_\tau^{*,-1} \int_0^\tau e^{-as} M_s^* \left(\nabla, \frac{\partial}{\partial y}\right) \eta_a(Z_r) \cdot (U_s dW_s, dB_s).$$

Hence

$$\begin{aligned}
&\int_M \left\langle E_y \left[e^{-a\tau} M_\tau \int_0^\tau e^{as} M_s^{-1} \frac{\partial}{\partial y}\omega_a(X_s, B_s) dB_s \middle| X_\tau = x \right], \eta(x) \right\rangle d\mu(x) \\
&= E_y \left[e^{-a\tau} M_\tau \int_0^\tau e^{as} M_s^{-1} \frac{\partial}{\partial y}\omega_a(X_s, B_s) dB_s, \eta(X_\tau) \right\rangle \Big] \\
&= I_1 + I_2,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= E_y \left[\left\langle e^{-a\tau} M_\tau \int_0^\tau e^{as} M_s^{-1} \frac{\partial}{\partial y}\omega_a(X_s, B_s) dB_s, e^{a\tau} M_\tau^{*,-1}\eta_a(X_0, B_0) \right\rangle \right], \\
I_2 &= E_y \Big[\Big\langle e^{-a\tau} M_\tau \int_0^\tau e^{as} M_s^{-1} \frac{\partial}{\partial y}\omega_a(X_s, B_s) dB_s, \\
&\qquad e^{a\tau} M_\tau^{*,-1} \int_0^\tau e^{-as} M_s^* (\nabla, \partial_y)\eta_a(X_s, B_s) \cdot (U_s dW_s, dB_s) \Big\rangle \Big].
\end{aligned}$$

Using the martingale property of the Itô integral, we have

$$\begin{aligned}
I_1 &= E_y \left[\left\langle \int_0^\tau e^{as} M_s^{-1} \frac{\partial}{\partial y}\omega_a(X_s, B_s) dB_s, \eta_a(X_0, B_0) \right\rangle \right] \\
&= E_y \left[\left\langle E \left[\int_0^\tau e^{as} M_s^{-1} \frac{\partial}{\partial y}\omega_a(X_s, B_s) dB_s \middle| (X_0, B_0) \right], \eta_a(X_0, B_0) \right\rangle \right] \\
&= 0.
\end{aligned}$$

On the other hand, using the L^2-isometry of the Itô integral, we have

$$\begin{aligned} I_2 &= E_y\left[\left\langle \int_0^\tau e^{as}M_s^{-1}\frac{\partial}{\partial y}\omega_a(X_s,B_s)dB_s, \int_0^\tau e^{-as}M_s^*(\nabla,\partial_y)\eta_a(X_s,B_s)\cdot(U_sdW_s,dB_s)\right\rangle\right] \\ &= E_y\left[\int_0^\tau \left\langle e^{as}M_s^{-1}\frac{\partial}{\partial y}\omega_a(X_s,B_s), e^{-as}M_s^*\frac{\partial}{\partial y}\eta_a(X_s,B_s)\right\rangle ds\right] \\ &= E_y\left[\int_0^\tau \left\langle \frac{\partial}{\partial y}\omega_a(X_s,B_s), \frac{\partial}{\partial y}\eta_a(X_s,B_s)\right\rangle ds\right]. \end{aligned}$$

The Green function of the background radiation process is given by $2(y\wedge z)$. Thus

$$\begin{aligned} &E_y\left[\int_0^\tau \left\langle \frac{\partial}{\partial y}\omega_a(X_s,B_s), \frac{\partial}{\partial y}\eta_a(X_s,B_s)\right\rangle ds\right] \\ &\qquad = 2\int_M\int_0^\infty (y\wedge z)\left\langle \frac{\partial}{\partial z}\omega_a(x,z), \frac{\partial}{\partial z}\eta_a(x,z)\right\rangle dzd\mu(x). \end{aligned}$$

By spectral decomposition, we have the Littelwood-Paley identity

$$\lim_{y\to\infty}\int_M\int_0^\infty (y\wedge z)\left\langle \frac{\partial}{\partial z}\omega_a(x,z), \frac{\partial}{\partial z}\eta_a(x,z)\right\rangle dzd\mu(x) = \int_M \langle \omega(x),\eta(x)\rangle d\mu(x).$$

Thus

$$\langle \omega,\eta\rangle_{L^2(\mu)} = 2\lim_{y\to\infty}\int_M \left\langle E_y\left[e^{-a\tau}M_\tau\int_0^\tau e^{as}M_s^{-1}\frac{\partial}{\partial y}\omega_a(X_s,B_s)dB_s\middle| X_\tau=x\right], \eta(x)\right\rangle d\mu(x).$$

This completes the proof of Theorem 2.2. □

The following martingale transform representation formula of the Riesz transforms on complete Riemannian manifolds, which is the extension of the Gundy-Varopoulos representation formula of the Riesz transforms on Euclidean space [5], is the correct reformulation of the one that we obtained in Theorem 3.2 in [6].

Theorem 2.3 *Let $R_a(L) = \nabla(a-L)^{-1/2}$. Then, for all $f\in C_0^\infty(M)$, we have*

$$R_a(L)f(x) = -2\lim_{y\to+\infty}E_y\left[e^{-a\tau}M_\tau\int_0^\tau e^{as}M_s^{-1}dQ_af(X_{s,s})dB_s\middle| X_\tau=x\right]. \tag{4}$$

In particular, in the case where $Ric+\nabla^2\phi=-a$, we have

$$R_a(L)f(x) = -2\lim_{y\to+\infty}E_y\left[U_\tau\int_0^\tau U_s^{-1}dQ_af(X_s,B_s)(U_sdW_s,dB_s)\middle| X_\tau=x\right]. \tag{5}$$

Proof. Applying Theorem 2.2 to $\omega = d(a-L)^{-1/2}f$, the proof of Theorem 2.3 is as the same as the one of Theorem 3.2 given in [6]. □

We now state the L^p-norm estimates of the Riesz transforms on complete Riemannian manifolds. Throughout this paper, for any $p\in(1,\infty)$, let

$$p^* = \max\left\{p,\ \frac{p}{p-1}\right\}.$$

The following result is a correction of Theorem 1.4 in [6].

Theorem 2.4 *Let M be a complete Riemannian manifold, and $\phi \in C^2(M)$. Then*
(i) for all $f \in C_0^\infty(M)$,

$$\|\nabla(a-L)^{-1/2}f\|_2 \le \|f\|_2, \tag{6}$$

(ii) if $Ric + \nabla^2\phi \equiv 0$, then for all $p \in (1,\infty)$,

$$\|\nabla(-L)^{-1/2}f\|_p \le 2(p^*-1)\|f\|_p, \quad \forall f \in C_0^\infty(M), \tag{7}$$

if $Ric + \nabla^2\phi \equiv -a$, where $a > 0$ is a constant, then for all $p \in (1,\infty)$,

$$\|\nabla(a-L)^{-1/2}f\|_p \le 2(p^*-1)(1+4\|T_1\|_p)\|f\|_p, \quad \forall f \in C_0^\infty(M), \tag{8}$$

where T_1 is the first exiting time of the standard 3-dimensional Brownian motion from the unit ball $B(0,1) = \{x \in \mathbb{R}^3 : \|x\| = 1\}$.
(iii) if $Ric + \nabla^2\phi \ge -a$, where $a \ge 0$ is a constant, then there is a numerical constant $C > 0$ such that for all $p > 1$,

$$\|\nabla(a-L)^{-1/2}f\|_p \le C(p^*-1)^{3/2}\|f\|_p, \quad \forall f \in C_0^\infty(M). \tag{9}$$

Proof. The case (i) for $p = 2$ is well known, cf. [6, 7]. By [6], for any fixed $x \in M$, there exists a bounded operator $A(x) \in \mathrm{End}(T_xM)$ such that that $d\omega(x) = A\nabla\omega(x)$ and $\|A(x)\|_{\mathrm{op}} \le 1$. In the case $Ric + \nabla^2\phi = -a$, we have

$$\nabla(a-L)^{-1/2}f(x) = -2\lim_{y\to+\infty} E_y\left[U_\tau \int_0^\tau U_s^{-1}A\nabla Q_af(X_s,B_s)dB_s \middle| X_\tau = x\right].$$

The stochastic integral in the above formula is a subordination of martingale transforms. By Burkholder's sharp L^p-inequality for martingale transforms [3] we obtain

$$\|\nabla(a-L)^{-1/2}f\|_p \le 2(p^*-1)\sup_{s\in[0,\tau]}\|A(X_s)\|_{\mathrm{op}}\left\|\int_0^\tau (\nabla,\partial_y)Q_a(f)(X_s,B_s)\cdot(U_sdW_s,dB_s)\right\|_p,$$

where $\|A(X_s)\|_{\mathrm{op}}$ denotes the operator norm of $A(X_s)$ on $T_{X_s}M$. Note that

$$\sup_{s\in[0,\tau]}\|A(X_s)\|_{\mathrm{op}} \le 1.$$

This yields

$$\|\nabla(a-L)^{-1/2}f\|_p \le 2(p^*-1)\left\|\int_0^\tau (\nabla,\partial_y)Q_a(f)(X_s,B_s)\cdot(U_sdW_s,dB_s)\right\|_p,$$

In [6], we have proved that, for all $1 < p < \infty$, it holds

$$\left\|\int_0^\tau (\nabla,\partial_y)Q_a(f)(X_s,B_s)\cdot(U_sdW_s,dB_s)\right\|_p \le (1+4\|T_1\|_p1_{a>0})\|f\|_p.$$

Combining this with the previous inequality, we obtain

$$\|\nabla(a-L)^{-1/2}f\|_p \le 2(p^*-1)(1+4\|T_1\|_p1_{a>0})\|f\|_p.$$

This proves the case of (ii).

In general case $Ric + \nabla^2\phi \geq -a$, we have

$$\nabla(a-L)^{-1/2}f(x) = 2\lim_{y\to+\infty} E_y\left[e^{-a\tau}M_\tau\int_0^\tau e^{as}M_s^{-1}A\nabla Q_a f(X_s,B_s)dB_s\middle|\, X_\tau = x\right].$$

By the L^p-contractivity of conditional expectation, see [6], we have

$$\|\nabla(a-L)^{-1/2}f\|_p \leq 2\lim\inf_{y\to\infty}\left\|e^{-a\tau}M_\tau\int_0^\tau e^{as}M_s^{-1}A\nabla Q_a f(X_s,B_s)dB_s\right\|_p.$$

Let

$$J_y = \left\{\int_0^\tau |\nabla Q_a f(X_s,B_s)|^2 ds\right\}^{1/2}.$$

By Theorem 2.6 due to Bañuelos and Baudoin in [2], under the condition $Ric + \nabla^2\phi \geq -a$, we can prove that

$$\left\|e^{-a\tau}M_\tau\int_0^\tau e^{as}M_s^{-1}dQ_a f(X_s,B_s)dB_s\right\|_p \leq 3\sqrt{p(2p-1)}\|J_y\|_p.$$

By Proposition 6.2 in our previous paper [7], for all $p\in(1,\infty)$, we proved that

$$\|J_y\|_p \leq B_p\|f\|_p,$$

where for all $p\in(1,2)$, $B_p = (2p)^{1/2}(p-1)^{-3/2}$, $B_2 = 1$, and for all $p\in(2,\infty)$, $B_p = \frac{p}{\sqrt{2(p-2)}}$. From the above estimates, for all $p\in(1,2)$, we can obtain

$$\begin{aligned}\|\nabla(a-L)^{-1/2}f\|_p &\leq 6\sqrt{2}p^{3/2}(2p-1)^{1/2}(p-1)^{-3/2}\|f\|_p\\ &\leq 12\sqrt{6}(p-1)^{-3/2}\|f\|_p,\end{aligned}$$

and for $p > 2$,

$$\begin{aligned}\|\nabla(a-L)^{-1/2}f\|_p &\leq 3\sqrt{2}p^{3/2}(2p-1)^{1/2}(p-2)^{-1/2}\|f\|_p\\ &\leq 6(p-1)^{3/2}(1+O(1/p))\|f\|_p.\end{aligned}$$

The proof of Theorem 2.4 is completed. □

Remark 2.5 The above proof corrects a gap in the proof of Theorem 1.4 given in [6] (p.270 line 9 to line 12 in [6]), where we used the Burkholder sharp L^p-inequality for martingale transforms. As $e^{-a\tau}M_\tau$ is not adapted with respect to the filtration $\mathcal{F}_s = \sigma(X_u, B_u, u\in[0,s])$, $s<\tau$, the proof given in [6] is valid only in the case $e^{-a\tau}M_\tau$ is independent of $(X_s : s\in[0,\tau])$, which only happens if $Ric + \nabla^2\phi \equiv -a$ for some constant $a\geq 0$..

The following result is the correction of Corollary 1.5 in [6].

Corollary 2.6 *Let M be a complete Riemannian manifold with non-negative Ricci curvature. Then there exists a numerical constant $C > 0$ such that for all $p > 1$,*

$$\|\nabla(-\Delta)^{-1/2}f\|_p \leq C(p^*-1)^{3/2}\|f\|_p.$$

In particular, if $Ric = 0$, i.e., if M is a Ricci flat Riemannian manifold, then for all $1 < p < \infty$,

$$\|\nabla(-\Delta)^{-1/2}f\|_p \leq 2(p^*-1)\|f\|_p.$$

In view of Theorem 2.4 and Corollary 2.6, we need to reformulate Conjecture 1.7 in [6] as follows.

Conjecture 2.7 *Let M be a complete Riemannian manifold, $\phi \in C^2(M)$. Suppose that $Ric(L) = Ric + \nabla^2\phi = 0$. Then there exists a constant $c > 0$ such that for all $p > 1$, we have*

$$c(p^* - 1)(1 + o(1)) \leq \|\nabla(-L)^{-1/2}\|_{p,p} \leq 2(p^* - 1).$$

In particular, on any complete Riemannian manifold M with flat Ricci curvature, for all $p > 1$, we have

$$c(p^* - 1)(1 + o(1)) \leq \|\nabla(-\Delta)^{-1/2}\|_{p,p} \leq 2(p^* - 1).$$

Remark 2.8 Using the Bellman function technique, Carbonaro and Dragičevič [4] proved that if $Ric + \nabla^2\phi \geq -a$, then for all $p \in (1, \infty)$,

$$\|\nabla(a - L)^{-1/2} f\|_p \leq 12(p^* - 1)\|f\|_p, \quad \forall f \in C_0^\infty(M).$$

It would be nice if one can find a probabilistic proof of this result.

3 Riesz transforms on Gaussian spaces

In this section, we give the proof of Corollary 1.6 in [6]. Let G be a compact Lie group endowed with a bi-invariant Riemannian metric, $\mathcal{G}$ its Lie algebra, and $n = \dim G$. Let $X_1, \ldots, X_n$ be an orthonormal basis of $\mathcal{G}$, and $\Delta_G = \sum\limits_{i=1}^{n} X_i^2$ the Laplace-Beltrami operator on G. In [1], Arcozzi proved that, the L^p-norm of the Riesz transform $R^G := \sum\limits_{i=1}^{n} R_{X_i} X_i$ on G satisfies $\|R^G\|_p \leq 2(p^* - 1)$ for all $p \in (1, \infty)$, where $R_{X_i} = X_i(-\Delta_G)^{-1/2}$ is the Riesz transform on G in the direction X_i. As the unit sphere S^{n-1} can be identified as $S^{n-1} = SO(n)/SO(n-1)$, where $SO(n)$ is the rotation group of $\mathbb{R}^n$, Arcozzi proved that the L^p-norm of the Riesz transform $R^{S^{n-1}} = \nabla^{S^{n-1}}(-\Delta_{S^{n-1}})^{-1/2}$ on S^{n-1} satisfies $\|R^{S^{n-1}}\|_p \leq 2(p^* - 1)$ for all $p \in (1, \infty)$. Let $S^{n-1}(\sqrt{n})$ be the $(n-1)$-dimensional sphere of radius $\sqrt{n}$. Then the L^p-norm of the Riesz transform $R^{S^{n-1}(\sqrt{n})}$ satisfies $\|R^{S^{n-1}(\sqrt{n})}\|_p \leq 2(p^* - 1)$. By the Poincaré limit, as $n \to \infty$, $S^{n-1}(\sqrt{n})$ endowed with the normalized volume measure converges in a proper way to the infinite dimensional Wiener space $\mathbb{R}^{\mathbb{N}}$ endowed with the Wiener measure, and the Laplace-Beltrami operator on $S^{n-1}(\sqrt{n})$ converges to the Orisntein-Uhlenbeck operator on $\mathbb{R}^{\mathbb{N}}$. From this, Arcozzi derived that the Riesz transform associated with the Ornstein-Uhlenbeck operator $L = \Delta - x \cdot \nabla$ on the Wiener space satisfies $\|\nabla(-L)^{-1/2}\|_p \leq 2(p^* - 1)$ for all $p \in (1, \infty)$.

In general, let $A \in M(n, \mathbb{R})$ be a positive definite symmetric matrix on $\mathbb{R}^n$, and let $\langle x, y\rangle_A = \langle x, Ay\rangle$, $\forall x, y \in \mathbb{R}^n$. Then $\sqrt{A} : (\mathbb{R}^n, \langle\cdot,\cdot\rangle) \to (\mathbb{R}^n, \langle\cdot,\cdot\rangle_A)$ is an isometry. Let $SO(n, A)$ be the rotation group on $(\mathbb{R}^n, \langle\cdot,\cdot\rangle_A)$, and S_A^{n-1} be the $(n-1)$-dimensional sphere in $(\mathbb{R}^n, \langle\cdot,\cdot\rangle_A)$. Then $S_A^{n-1} = SO(n, A)/SO(n-1, A)$. By the same argument as used by Arcozzi [1], we can prove that the L^p-norm of the Riesz transform on $SO(n, A)$ satisfies $\|R^{SO(n,A)}\|_p \leq 2(p^* - 1)$, and the L^p-norm of the Riesz transform on S_A^{n-1} satisfies $\|R^{S_A^{n-1}}\|_p \leq 2(p^* - 1)$. Similarly, we have $\|R^{S_A^{n-1}(\sqrt{n})}\|_p \leq 2(p^* - 1)$. Thus, we have proved the following

Theorem 3.1 *Let $A \in M(n, \mathbb{R})$ be a positive definite symmetric matrix on $\mathbb{R}^n$, and let*

$$L_A = \Delta - Ax \cdot \nabla$$

be the Ornstein-Uhlenbeck operator on the Gaussian space $(\mathbb{R}^n, \mu_A)$, where

$$d\mu_A(x) = \frac{1}{(2\pi \ \det A)^{n/2}} e^{-\langle x, Ax \rangle} dx.$$

Then, for all $1 < p < \infty$, the L^p-norm of the Riesz transform $R = \nabla(-L_A)^{-1/2}$ on $(\mathbb{R}^n, \mu_A)$ satisfies

$$\|\nabla(-L_A)^{-1/2}\|_p \leq 2(p^* - 1).$$

Using the Poincaré limit, we can derive the following result from Theorem 3.1.

Theorem 3.2 *(i.e., Corollary 1.6 in [6]) Let (W, H, μ_A) be an abstract Wiener space, where W is a real separable Banach space, H is a real separable Hilbert space which is densely embedded in W, $A \in \mathcal{L}(H)$ be a self-adjoint positive definite operator with finite Hilbert-Schmidt norm, and μ the Gaussian measure on W with mean zero and with covariance A. Let*

$$L_A = \Delta - Ax \cdot \nabla$$

be the generalized Ornstein-Uhlenbeck operator on (W, H, μ_A). Then, for all $1 < p < \infty$, the L^p-norm of the Riesz transform $R = \nabla(-L_A)^{-1/2}$ on (W, H, μ_A) satisfies

$$\|\nabla(-L_A)^{-1/2}\|_p \leq 2(p^* - 1).$$

4 Beurling-Ahlfors transforms

Throughout this section, let M be a complete and stochastically complete Riemannian manifold, $n = \dim M$. Let X_t be Brownian motion on M, W_k the k-th Weitzenböck curvature operator. Let $A_i \in \mathrm{End}(\Lambda^k T^* M)$, $i = 1, 2$, be the bounded endomorphism which, in a local normal coordinate $(e_1, \ldots, e_n)$ at any fixed point x, is defined by

$$A_1 = (a_i a_j^*)_{n \times n}, \quad A_2 = (a_i^* a_j)_{n \times n},$$

where $a_i = \mathrm{int}_{e_i}$ is the inner multiplication by e_i, and $a_j^* = e_j^* \wedge$ is the exterior multiplication by e_j, $i, j = 1, \ldots, n$. For details, see [9].

Let $M_t \in \mathrm{End}(\Lambda^k T^*_{X_0} M, \Lambda^k T^*_{X_t} M)$ be defined by

$$\frac{\nabla M_t}{\partial t} = -W_k(X_t) M_t, \quad M_0 = \mathrm{Id}_{\Lambda^k T^*_{X_0} M}.$$

For any fixed $T > 0$, the backward heat semigroup generated by the Hodge Laplacian $\Box$ on k-forms is defined by

$$\omega(x, T - s) = e^{-(T-s)\Box} \omega(x), \quad \forall x \in M, s \in [0, T], \ \omega \in C_0^\infty(\Lambda^k T^* M).$$

Recall that, the Weitzenböck formula reads as follows

$$\Box = -\mathrm{Tr} \nabla^2 + W_k.$$

We now state the martingale transform representation formula for the Beurling-Ahlfors transforms on k-forms over complete Riemannian manifolds .

Theorem 4.1 *Let M be a complete and stochastically complete Riemannian manifold. Suppose that $W_k \geq -a$, where $a \geq 0$ is a constant. Then, for all $\omega, \eta \in C_0^\infty(\Lambda^k T^*M)$, we have*

$$\langle dd^*(a+\Box)^{-1}\omega, \eta\rangle = 2\lim_{T\to\infty}\int_M \langle S^T_{A_2}\omega, \eta\rangle dx,$$

$$\langle d^*d(a+\Box)^{-1}\omega, \eta\rangle = 2\lim_{T\to\infty}\int_M \langle S^T_{A_1}\omega, \eta\rangle dx,$$

where, for a.s. $x \in M$,

$$S^T_{A_i}\omega(x) = E\left[M_T e^{-aT}\int_0^T e^{at}M_t^{-1}A_i\nabla\omega_a(X_t, T-t)dX_t\middle| X_T = x\right], \quad i = 1, 2.$$

In particular, the Beurling-Ahlfors transform

$$S_B\omega := (d^*d - dd^*)(a+\Box)^{-1}\omega$$

has the following martingale transform representation: for a.s. $x \in M$,

$$S_B\omega(x) = 2\lim_{T\to\infty} E\left[M_T e^{-aT}\int_0^T e^{at}M_t^{-1}B\nabla\omega_a(X_t, T-t)dX_t\middle| X_T = x\right],$$

where

$$B = A_1 - A_2.$$

Remark 4.2 The martingale transform representation formulas in Theorem 4.1 are the correct reformulation of the formulas that we obtained in Theorem 3.4 in [9], where the martingale transform representation formulas of S_{A_i} and S_B were given in the following way

$$S^T_{A_i}\omega(x) = E\left[\int_0^T e^{a(t-T)}M_T M_t^{-1}A_i\nabla\omega_a(X_t, T-t)dX_t\middle| X_T = x\right], \quad i = 1, 2,$$

and

$$S_B\omega(x) = 2\lim_{T\to\infty} E\left[\int_0^T e^{a(t-T)}M_T M_t^{-1}B\nabla\omega_a(X_t, T-t)dX_t\middle| X_T = x\right].$$

The same correction should also be made for Theorem 3.5 in [9], where $a = 0$. The reason is that, as pointed out by Bañuelos and Baudoin in [2], M_T is not adapted with respect to the filtration $\mathcal{F}_t = \sigma(X_s : s \in [0, t])$, $t < T$. Moreover, in the proof of Theorem 1.2 in [9] (p.135, line 7 to line 8), we used the Burkholder-Davis-Gundy inequality to derive that

$$\|S^T_{A_i}\omega\|_p \leq C_p \sup_{0\leq t\leq T}\|e^{a(t-T)}M_T M_t^{-1}A_i\|_{\rm op}\left\|\left\{\int_0^T |\overline{\nabla}\omega_a(X_t, T-t)|^2 dt\right\}^{1/2}\right\|_p,$$

where $\|\cdot\|_{\rm op}$ denotes the operator norm, and C_p is a constant. However, except that M_T is independent of the $(X_t : t \in [0, T])$, one cannot use the Burkholder-Davis-Gundy inequality in above way, due to the fact that M_T is not adapted with respect to the filtration $\mathcal{F}_t = \sigma(X_s : s \in [0, t])$, $t < T$.

Proof of Theorem 4.1. By Remark 4.2, we need only to correct the martingale transform representation formulas appeared in Theorem 3.4 and Theorem 3.5 in [9] in the right way stated in Theorem 4.1. Thus, the original proof given in [9] for these formulas remain valid after a small modification. To save the length of the paper, we omit it here. $\square$

Proposition 4.3 *For all constant $a \geq 0$ and $\omega \in C_0^\infty(\Lambda^k T^*M)$, we have*

$$\|dd^*(a+\square)^{-1}\omega\|_2^2 + \|d^*d(a+\square)^{-1}\omega\|_2^2 = \|\square(a+\square)^{-1}\omega\|_2^2, \tag{10}$$

Moreover,

$$\begin{aligned}\|dd^*(a+\square)^{-1}\omega\|_2 &\leq \|\omega\|_2,\\ \|d^*d(a+\square)^{-1}\omega\|_2 &\leq \|\omega\|_2,\end{aligned}$$

and

$$\|(d^*d - dd^*)(a+\square)^{-1}\omega\|_2 \leq 2\|\omega\|_2.$$

Proof. By Gaffney's integration by parts formula, we have

$$\begin{aligned}\|dd^*(a+\square)^{-1}\omega\|_2^2 &= \int_M \langle dd^*(a+\square)^{-1}\omega, dd^*(a+\square)^{-1}\omega\rangle dv\\ &= \int_M \langle (a+\square)^{-1}\omega, dd^*dd^*(a+\square)^{-1}\omega\rangle dv.\end{aligned}$$

Similarly, we can prove

$$\|d^*d(a+\square)^{-1}\omega\|_2^2 = \int_M \langle (a+\square)^{-1}\omega, d^*dd^*d(a+\square)^{-1}\omega\rangle dv.$$

Using the fact that $dd^*dd^* + d^*dd^*d = \square^2$, we get

$$\|dd^*(a+\square)^{-1}\omega\|_2^2 + \|d^*d(a+\square)^{-1}\omega\|_2^2 = \int_M \langle (a+\square)^{-1}\omega, \square^2(a+\square)^{-1}\omega\rangle dv.$$

This proves the identity (10). Again, integration by parts yields

$$\begin{aligned}\|(a+\square)\omega\|_2^2 &= \|\square\omega\|_2^2 + 2a\langle\langle \omega, \square\omega\rangle\rangle + a^2\|\omega\|_2^2\\ &= \|\square\omega\|_2^2 + 2a\|d\omega\|_2^2 + 2a\|d^*\omega\|_2^2 + a^2\|\omega\|_2^2\\ &\geq \|\square\omega\|_2^2,\end{aligned}$$

which implies that

$$\|\square(a+\square)^{-1}\omega\|_2 \leq \|\omega\|_2. \tag{11}$$

Combining (10) with (11), we obtain

$$\|dd^*(a+\square)^{-1}\omega\|_2^2 + \|d^*d(a+\square)^{-1}\omega\|_2^2 \leq \|\omega\|_2^2.$$

This finishes the proof of Proposition 4.3. $\square$

We now state the L^p-norm estimates of the Beurling-Ahlfors transforms on complete Riemannian manifolds. The following result is the restatement of Theorem 1.2 and Theorem 5.1 in [9]. Here, as in [9], $\|\cdot\|_{\mathrm{op}}$ denotes the operator norm.

Theorem 4.4 *Suppose that there exists a constant $a \geq 0$ such that*

$$W_k \geq -a.$$

Then, there exists a universal constant $C > 0$ such that for all $1 < p < \infty$, and for all $\omega \in C_0^\infty(\Lambda^k T^* M)$,

$$\|S_{A_i}\omega\|_p \leq C(p^*-1)^{3/2}\|A_i\|_{\rm op}\|\omega\|_p,$$

and

$$\|S_B\omega\|_p \leq C(p^*-1)^{3/2}\|B\|_{\rm op}\|\omega\|_p.$$

In particular, in the case where $W_k \equiv -a$, we have

$$\|S_{A_i}\omega\|_p \leq 2(p^*-1)\|A_i\|_{\rm op}\|\omega\|_p,$$

and

$$\|S_B\omega\|_p \leq 2(p^*-1)\|B\|_{\rm op}\|\omega\|_p.$$

Proof. By Proposition 4.3, we need only to study the case $p \neq 2$. For simplicity, we only consider the case $W_k \geq 0$. The general case $W_k \geq -a$ can be similarly proved. Let

$$Z_t^i = M_t \int_0^t M_s^{-1} A_i \nabla\omega(X_s, t-s)dX_s, \quad i = 1, 2.$$

By Theorem 2.6 due to Bañuelos and Baudoin in [2], for all $p \in (1, \infty)$, we have

$$\|Z_T^i\|_p \leq 3\sqrt{p(2p-1)}\left\|\left(\int_0^T |A_i\nabla\omega(X_t, T-t)|^2 dt\right)^{\frac{1}{2}}\right\|_p.$$

Obviously, we have

$$\left\|\left(\int_0^T |A_i\nabla\omega(X_t, T-t)|^2 dt\right)^{\frac{1}{2}}\right\|_p \leq \|A_i\|_{\rm op}\left\|\left(\int_0^T |\nabla\omega(X_t, T-t)|^2 dt\right)^{\frac{1}{2}}\right\|_p.$$

By the same argument as used in the proofs of Proposition 6.2 and Proposition 6.3 in [7], for all $1 < p < \infty$, we can prove that

$$\left\|\left(\int_0^T |\nabla\omega(X_t, T-t)|^2 dt\right)^{\frac{1}{2}}\right\|_p \leq B_p\|\omega\|_p,$$

where $B_p = (2p)^{1/2}(p-1)^{-3/2}$ for $p \in (1, 2)$, $B_p = 1$ for $p = 2$, and $B_p = \frac{p}{\sqrt{2(p-2)}}$ if $p > 2$. Hence, for $1 < p < 2$,

$$\begin{aligned}\|S_{A_i}^T\omega\|_p &\leq 3\sqrt{p(2p-1)}\|A_i\|_{\rm op}\frac{(2p)^{1/2}}{(p-1)^{3/2}}\|\omega\|_p \\ &\leq 6\sqrt{6}(p-1)^{-3/2}\|A_i\|_{\rm op}\|\omega\|_p,\end{aligned}$$

and for $p > 2$,

$$\|S_{A_i}^T \omega\|_p \leq 3(p-1)^{3/2}(1 + O((p-1)^{-1}))\|A_i\|_{\mathrm{op}}\|\omega\|_p.$$

Indeed, by duality argument as used in [9], for all $p > 2$, we have

$$\|S_{A_i}^T\|_{p,p} = \|S_{A_i}^T\|_{q,q},$$

which yields for $p > 2$,

$$\|S_{A_i}^T \omega\|_p \leq 6\sqrt{6}(p-1)^{3/2}\|A_i\|_{\mathrm{op}}\|\omega\|_p.$$

In summary, for all $1 < p < \infty$, we have proved that

$$\|S_{A_i}^T \omega\|_p \leq 6\sqrt{6}(p^*-1)^{3/2}\|A_i\|_{\mathrm{op}}\|\omega\|_p.$$

Similarly, for all $1 < p < \infty$, we can prove

$$\|S_{B}^T \omega\|_p \leq 6\sqrt{6}(p^*-1)^{3/2}\|B\|_{\mathrm{op}}\|\omega\|_p.$$

In the particular case where $W_k \equiv -a$, we have

$$S_{A_i}^T \omega(x) = E\left[U_T \int_0^T U_t^{-1} A_i \nabla \omega_a(X_t, T-t) dX_t \middle| X_T = x \right], \quad i = 1, 2, \ \text{a.s.} x \in M.$$

The L^p-contractiveness of the conditional expectation yields

$$\begin{aligned} \|S_{A_i}^T \omega\| &= \left\| U_T \int_0^T U_t^{-1} A_i \nabla \omega_a(X_t, T-t) dX_t \right\|_p \\ &= \left\| \int_0^T U_t^{-1} A_i \nabla \omega_a(X_t, T-t) dX_t \right\|_p. \end{aligned}$$

Using the Burkholder sharp L^p-inequality for martingale transforms, for all $p > 1$, we deduce that

$$\|S_{A_i}^T \omega\| \leq (p^*-1) \sup_{0 \leq t \leq T} \|U_t^{-1} A_i U_t\|_{\mathrm{op}} \left\| \int_0^T U_t^{-1} \nabla \omega_a(X_t, T-t) dX_t \right\|_p. \tag{12}$$

By Itô's formula, we can prove that (see Eq. (49) in [9])

$$\omega(X_T) - U_T \omega_a(X_0, T) = U_T \int_0^T U_t^{-1} \nabla \omega_a(X_t, T-t) dX_t. \tag{13}$$

Substituting (13) into (12), we have

$$\|S_{A_i}^T \omega\|_p \leq (p^*-1)\|A_i\|_{\mathrm{op}} \|\omega(X_T) - U_T \omega_a(X_0, T)\|_p.$$

Using the argument in [9], we obtain

$$\|S_A^T \omega\|_p \leq (p^*-1)\left(1 + e^{-2\min\{\frac{1}{p}, 1-\frac{1}{p}\}aT}\right)\|\omega\|_p.$$

Hence

$$\|dd^*(a+\Box)^{-1}\omega\|_p \le 2\lim_{T\to\infty}\|S^T_{A_1}\omega\|_p \le 2(p^*-1)\|A_1\|_{\rm op}\|\omega\|_p,$$

$$\|d^*d(a+\Box)^{-1}\omega\|_p \le 2\lim_{T\to\infty}\|S^T_{A_2}\omega\|_p \le 2(p^*-1)\|A_2\|_{\rm op}\|\omega\|_p,$$

and

$$\|(dd^*-d^*d)(a+\Box)^{-1}\omega\|_p \le 2\lim_{T\to\infty}\|S^T_{B}\omega\|_p \le 2(p^*-1)\|B\|_{\rm op}\|\omega\|_p.$$

The proof of Theorem 4.4 is completed. $\Box$

Remark 4.5 The above proof corrects a gap contained in [9]. In summary, the L^p-norm estimates in Theorem 4.4 indicates that the results in Theorem 1.2, Theorem 1.3 and Theorem 1.4, Theorem 5.1 and Corollary 5.2 obtained in [9] remain valid. As a consequence, the main theorems proved in [9] remain valid. In particular, see Theorem 1.3 in [9], on complete and stochastically complete Riemannian manifolds non-negative Weitzenböck curvature operator $W_k \ge 0$, where $1 \le k \le n = {\rm dim}M$, the Weak L^p-Hodge decomposition theorem holds for k-forms, the De Rham projection $\mathrm{P}_1 = dd^*\Box^{-1}$, the Leray projection $\mathrm{P}_2 = d^*d\Box^{-1}$ and the Beurling-Ahlfors transform $B_k = (d^*d - dd^*)\Box^{-1}$ on k-form is bounded in L^p for all $1 < p < \infty$.

5 Time reversal martingale transformation representation formula for the Riesz transfroms

In this section, we prove a time reversal martingale transformation representation formula for the Riesz transforms on complete Riemannian manifolds.

First, we prove the following time reversal martingale transformation representation formula for one forms.

Theorem 5.1 *Let $\widehat{X}_t = X_{\tau-t}$, and $\widehat{B}_t = B_{\tau-t}$, $t \in [0,\tau]$. Let $\widehat{M_t}$ be the solution to the covariant SDE*

$$\begin{aligned}\frac{\nabla}{\partial t}\widehat{M_t} &= -\widehat{M_t}(Ric + \nabla^2\phi)(\widehat{X}_t),\\ \widehat{M_0} &= \mathrm{Id}_{T_{\widehat{X}_0}M}.\end{aligned}$$

For any $\omega \in C_0^\infty(\Lambda^1 T^*M)$, let $\omega_a(x,y) = e^{-y\sqrt{a+\Box_\phi}}\omega(x)$, $\forall x \in M, y \ge 0$. Then, for a.s. $x \in M$,

$$\frac{1}{2}\omega(x) = \lim_{y\to+\infty} E_y\left[\widehat{Z}_\tau \middle| \widehat{X}_0 = x\right],$$

where

$$\widehat{Z}_\tau = \int_0^\tau e^{-at}\widehat{M_t}\partial_y\omega_a(\widehat{X}_t,\widehat{B}_t)d\widehat{B}_t - \int_0^\tau e^{-at}\widehat{M_t}\partial_y^2\omega(\widehat{X}_t,\widehat{B}_t)dt.$$

Proof. By Theorem 2.2, we have

$$\frac{1}{2}\omega(x) \quad = \quad \lim_{y\to+\infty} E_y\left[Z_\tau \middle| X_\tau = x\right],$$

where

$$Z_\tau = e^{-a\tau} M_\tau \int_0^\tau e^{as} M_s^{-1} \nabla_y \omega_a(X_s, B_s) dB_s.$$

Taking $0 = s_0 < s_1 < \ldots < s_n < s_{n+1} = \tau$ be a partition of $[0,\tau]$, then

$$Z_{\tau,n} := e^{-a\tau} M_\tau \sum_{i=1}^{N} e^{as_i} M_{s_i}^{-1} \nabla_y \omega(X_{s_i}, B_{s_i})(B_{s_{i+1}} - B_{s_i})$$

converges in L^2 and in probability to Z_τ. We can rewrite $Z_{\tau,n}$ as follows

$$Z_{\tau,n} = \sum_{i=1}^{N} e^{-a(\tau-s_i)} M_\tau M_{s_i}^{-1} \nabla_y \omega(X_{s_i}, B_{s_i})(B_{s_{i+1}} - B_{s_i}).$$

Note that

$$\begin{aligned}
\partial_s \widehat{M}_{\tau-s} &= \widehat{M}_{\tau-s} Ric(L)(\widehat{X}_{\tau-s}) \\
&= \widehat{M}_{\tau-s} Ric(L)(X_s),
\end{aligned}$$

and

$$\begin{aligned}
\partial_s(M_\tau M_s^{-1}) &= -M_\tau M_s^{-1} \partial_s M_s M_s^{-1} \\
&= M_\tau M_s^{-1} Ric(L)(X_s) M_s M_s^{-1} \\
&= (M_\tau M_s^{-1}) Ric(L)(X_s).
\end{aligned}$$

By the uniqueness of the solution to ODE, as $\widehat{M}_{\tau-s}\Big|_{s=\tau} = M_\tau M_s^{-1}\big|_{s=\tau} = \mathrm{Id}_{T_{\widehat{X}_0} M}$, we have

$$M_\tau M_s^{-1} = \widehat{M}_{\tau-s}.$$

Therefore

$$Z_{\tau,n} = \sum_{i=1}^{N} e^{-a(\tau-s_i)} \widehat{M}_{\tau-s_i} \nabla_y \omega(\widehat{X}_{\tau-s_i}, \widehat{B}_{\tau-s_i})(\widehat{B}_{\tau-s_{i+1}} - \widehat{B}_{\tau-s_i})$$

Let $t_i = \tau - s_i$. Then $\tau = t_0 > t_1 > \ldots > t_n > t_{n+1} = 0$, and

$$Z_{\tau,n} = \sum_{i=1}^{N} e^{-at_i} \widehat{M}_{t_i} \nabla_y \omega(\widehat{X}_{t_i}, \widehat{B}_{t_i})(\widehat{B}_{t_{i+1}} - \widehat{B}_{t_{i+1}}).$$

By Taylor's formula, we have

$$\omega(\widehat{X}_{t_i}, \widehat{B}_{t_i}) = \omega(\widehat{X}_{t_{i+1}}, \widehat{B}_{t_{i+1}}) - \nabla_y \omega(\widehat{X}_{t_{i+1}}, \widehat{B}_{t_{i+1}})(\widehat{B}_{t_{i+1}} - \widehat{B}_{t_i}) + O\left((\widehat{B}_{t_{i+1}} - \widehat{B}_{t_i})^2\right).$$

Hence

$$\begin{aligned} Z_{\tau,n} &= \sum_{i=1}^{N} e^{-at_i}\widehat{M}_{t_i}\nabla_y\omega(\widehat{X}_{t_{i+1}},\widehat{B}_{t_{i+1}})(\widehat{B}_{t_{i+1}}-\widehat{B}_{t_i}) \\ &\quad -\sum_{i=1}^{N} e^{-at_i}\widehat{M}_{t_i}\nabla_y^2\omega(\widehat{X}_{t_{i+1}},\widehat{B}_{t_{i+1}})(\widehat{B}_{t_{i+1}}-\widehat{B}_{t_i})^2+O\left((\widehat{B}_{t_{i+1}}-\widehat{B}_{t_i})^3\right). \end{aligned}$$

which converges in L^2 and in probability to the following limit

$$\widehat{Z}_\tau=\int_0^\tau e^{-at}\widehat{M}_t\nabla_y\omega(\widehat{X}_t,\widehat{B}_t)d\widehat{B}_t-\int_0^\tau e^{-at}\widehat{M}_t\nabla_y^2\omega(\widehat{X}_t,\widehat{B}_t)dt.$$

The proof of Theorem 5.1 is completed. □

By Theorem 5.1, we can prove the following time reversal martingale transformation representation formula for the Riesz transforms on complete Riemannian manifolds.

Theorem 5.2 *Let $R_a(L)=\nabla(a-L)^{-1/2}$. Then, for $f\in C_0^\infty(M)$, we have*

$$R_a(L)f(x)=-2\lim_{y\to+\infty}E_y\left[\widehat{Z}_\tau\middle|\widehat{X}_0=x\right],$$

where

$$\widehat{Z}_\tau=\int_0^\tau e^{-as}\widehat{M}_s dQ_af(\widehat{X}_s,\widehat{B}_s)d\widehat{B}_s-\int_0^\tau e^{-as}\widehat{M}_s\partial_y dQ_af(\widehat{X}_s,\widehat{B}_s)ds.$$

Remark 5.3 As noticed in [6], there exists a standard one dimensional Brownian motion β_t such that

$$d\widehat{B}_t=d\beta_t+\frac{dt}{\widehat{B}_t},\quad t\in(0,\tau].$$

6 Time reversal martingale transforms representation formula for the Beurling-Ahlfors transforms

Similarly to the proof of Theorem 5.1, we prove a time reversal martingale transformation representation formula for the Beurling-Ahlfors transforms on complete Riemannian manifolds.

Theorem 6.1 *Let $\widehat{X}_t=X_{T-t}$, $t\in[0,T]$. Let $\widehat{M}_t$ be the solution to the covariant equation*

$$\frac{\nabla\widehat{M}_t}{\partial t}=-\widehat{M}_tW_k(\widehat{X}_t),\quad \widehat{M}_0=\mathrm{Id}_{\Lambda^kT^*_{\widehat{X}_0}M}.$$

Then, for any $\omega\in C_0^\infty(\Lambda^1T^*M)$, the Beurling-Ahlfors transform

$$S_B\omega:=(d^*d-dd^*)(a+\square)^{-1}\omega$$

has the following time reversal martingale transform representation: for a.s. $x\in M$,

$$S_B\omega(x)=2\lim_{T\to\infty}E\left[\widehat{Z}_T\middle|\widehat{X}_0=x\right],$$

where

$$\widehat{Z}_T = \int_0^T e^{-as}\widehat{M}_s B\nabla\omega_a(\widehat{X}_s, s)d\widehat{X}_s - \int_0^T e^{-as}\widehat{M}_s B\mathrm{Tr}\nabla^2\omega_a(\widehat{X}_s, s)ds.$$

To end this paper, let us mention that, in a forthcoming paper [10], we will prove a martingale transform representation formula for the Riesz transforms associated with the Dirac operator acting on Hermitian vector bundles over complete Riemannian manifolds and for the Riesz transforms associated with the $\bar{\partial}$-operator acting on holomorphic Hermitian vector bundles over complete Kähler manifolds. By the same argument as used in this paper, we can prove some explicit dimension free L^p-norm estimates of these Riesz transforms on complete Riemannian or Kähler manifolds with suitable curvature conditions. See also [8].

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